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On the study of strongly parabolic problems involving anisotropic operators in L^1

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Abstract

This paper is concerned with the study of the nonlinear Dirichlet parabolic problem in a bounded subset $\Omega \subset I\!R^N$

$$u_t + Au + g(x, t, u, \nabla u) = f - \operatorname{div} \phi(u),$$

where A is an operator of Leray-Lions type acted from the parabolic anisotropic space $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ into its dual. g is a nonlinear term having a growth condition with respect to ∇u and satisfying a sign condition with no growth condition with respect to u. In addition, when the initial condition u_0 and the data f are assumed to be merely integrable and $\phi(\cdot) \in C^0(I\!R, I\!R^N)$, we prove the existence of entropy solutions for this class of problems.

Keywords Anisotropic Sobolev space · Nonlinear parabolic problems · Entropy solution

Mathematics Subject Classification 35K55 · 46E30 · 46E35 · 46E36 · 47J05

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1 Introduction

Let $T > 0$ and $Q_T = \Omega \times (0, T)$ be a cylinder over the open bounded domain $\Omega \subset I\!R^N$ $(N \geq 2)$. We denote by Σ_T the lateral surface $\partial\Omega \times (0, T)$.

Boccardo, Gallouët and Vazquez have studied in [6] the following nonlinear parabolic equation

$$\begin{cases} u_t + Au + \alpha_0 |u|^{s-1} u = f & \text{in} \quad Q_T, \\ u \ = 0 & \text{on} \quad \Sigma_T \\ u(x, 0) = 0 & \text{in} \quad \Omega, \end{cases}$$

where $\alpha_0 > 0$ and $Au = -\text{div}\,(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian operator with $p > 1 + \frac{N}{N+1}$. When the data f is in $L^1(Q_T)$ and $s > \frac{p(N+1)-N}{N}$, the authors have established the existence and regularity of solutions for such problems.

In [8], Boccardo, Dall'Aglio, Gallouët and Orsina proved the existence result for the nonlinear parabolic Cauchy-Dirichlet problem

$$\begin{cases} u_t - Au = \mu & \text{in} \quad Q_T, \\ u \ = 0 & \text{on} \quad \Sigma_T, \\ u(x, 0) = 0 & \text{in} \quad \Omega, \end{cases}$$

where $Au = \text{div}\, a(x, t, u, \nabla u)$ is a classical divergence operator of Leray-Lions type and μ belongs to $M(Q_T)$, the space of bounded Borel measures on Q_T.

The case of isotropic elliptic and parabolic equations for which the principal part of the operator behaves like the Leray-Lions operator, has been the subject of numerous studies, we can cite, among others, the references [5,6,8,12,20], where the authors obtained existence of solutions by considering lower order terms with quadratic growth or subquadratic growth with respect to the gradient.

Regarding the anisotropic case, let us point out that anisotropic spaces involving anisotropic exponent $\vec{p} = (p_i)_{i=1,\ldots,N}$ are the appropriate framework to deal with a class of problems having non-standard structural conditions, one prototype of the differential operator considered is the $\vec{p}-$Laplacian

$$\Delta_{\vec{p}}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u),$$

which generalizes the $p-$Laplace operator. It is not a surprise that new difficulties occur in the anisotropic spaces. To overcome these difficulties, we combine the classical techniques with recent ones that appeared when treating anisotropic problems, we refer the reader to the papers [1,3,13,15–18] where existence results are obtained for various types of nonlinear anisotropic elliptic and parabolic equations with respect to the data.

Let us also mention that in recent years much attention has been given to the study of anisotropic variable exponent Sobolev spaces and several studies have been devoted to the investigation of related problems. This attention comes essentially from their applications in the study of nonhomogeneous materials that behave differently on different space directions, we can refer here to the electrorheological and thermoelectric fluids (see for example [2]).

The aim of this paper is to study the parabolic generalization of a class of nonlinear elliptic problems considered earlier in both isotropic and anisotropic cases using a compactness result. More precisely, we establish the existence of entropy solutions for some strongly nonlinear anisotropic parabolic problem of the form

$$\begin{cases} u_t + Au + g(x,t,u,\nabla u) = f - \operatorname{div} \phi(u) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \Sigma_T, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in L^1(\Omega)$ and the right hand side is assumed to satisfy

$$f \in L^1(Q_T) \text{ and } \phi(.) = (\phi_1(.), \ldots, \phi_N(.)) \in C^0(I\!R, I\!R^N).$$

The Leray-Lions operator A acted from $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ into its dual $L^{\vec{p}'}(0,T; W^{-1,\vec{p}'}(\Omega))$ (spaces as defined in Sect. 2) is given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,t,\nabla u) + d(x,t)|u|^{p_0-2}u,$$

where $d(x,t)$ is a positive function in $L^\infty(Q_T)$ such that there exists a constant $d_0 > 0$ with $d(x,t) \geq d_0$ a.e in Q_T, while $(a_i(x,t,\xi))_{i=1,\ldots,N}$ are Carathéodory functions (measurable with respect to (x,t) in Q_T for every ξ in $I\!R^N$, and continuous with respect to ξ in $I\!R^N$ for almost every (x,t) in Q_T) satisfying

$$|a_i(x,t,\xi)| \leq \beta(K_i(x,t) + |\xi_i|^{p_i-1}) \qquad \text{for} \quad i = 1,\ldots,N \tag{1.2}$$

$$a_i(x,t,\xi)\xi_i \geq \alpha|\xi_i|^{p_i} \qquad \text{for} \quad i = 1,\ldots,N. \tag{1.3}$$

For a.e. $(x,t) \in Q_T$, for every $\xi, \xi' \in I\!R^N$, $\xi \neq \xi'$

$$(a_i(x,t,\xi) - a_i(x,t,\xi'))(\xi_i - \xi_i') > 0 \qquad \text{for} \quad i = 1,\ldots,N. \tag{1.4}$$

Here $K_i(x,t)$ is a nonnegative function lying in $L^{p_i'}(Q_T)$ and $\alpha, \beta > 0$.
As regards the strongly nonlinear perturbation lower-order term, we assume that $g(x,s,\xi)$ has no growth conditions with respect to u, and satisfies the following classical sign condition and natural growth on ∇u : $g : \Omega \times (0,T) \times I\!R \times I\!R^N \mapsto I\!R$ is a Carathéodory function, such that for a.e. $(x,t) \in Q_T$, all $(s,\xi) \in I\!R \times I\!R^N$

$$g(x,t,s,\xi).s \;\geq\; 0, \tag{1.5}$$

$$|g(x,t,s,\xi)| \leq b(|s|)\left(c(x,t) + \sum_{i=1}^{N} |\xi_i|^{p_i}\right), \tag{1.6}$$

where $b(\cdot) : I\!R^+ \longmapsto I\!R^+$ is a continuous nondecreasing function, and $c(\cdot,\cdot) : \Omega \times (0,T) \longmapsto I\!R^+$ with $c(\cdot,\cdot) \in L^1(Q_T)$.

Herein $p_0, p_1, \cdots, p_N$ are $N+1$ real positive numbers such that $\frac{2N}{N+2} < p_i < \infty$ for any $i = 0, 1, \ldots, N$.

This paper is organized as follows. In Sect. 2, we introduce some definitions and results concerning anisotropic parabolic spaces. Section 3 contains some technical lemmas needed to establish our main results. The last section will be devoted to prove the existence of entropy solutions for our parabolic problem in the anisotropic spaces.

2 Parabolic anisotropic spaces

Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$ with boundary $\partial\Omega$.

Let $p_0, p_1, \ldots, p_N$ be $N+1$ real constants numbers, with $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$.

We denote by

$$\vec{p} = \{p_0, p_1, \ldots, p_N\}, \ \underline{p} = \min\{p_i, \ i = 0, 1, 2, \ldots, N\} \text{ and}$$
$$D^0 u = u, \ D^i u = \frac{\partial u}{\partial x_i}, \ i = 1, \ldots, N.$$

We introduce the anisotropic Sobolev space

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\},$$

under the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

We also define $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (2.1), where $\mathcal{C}_0^\infty(\Omega)$ is the space of all continuous functions with compact support in Ω, that have continuous derivatives for any order.

The dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p}'}(\Omega)$, where $\vec{p}' = \{p'_0, p'_1, \ldots, p'_N\}$, $\frac{1}{p'_i} + \frac{1}{p_i} = 1$, $i = 0, 1, \ldots, N$, i.e.,

$$\forall F \in W^{-1,\vec{p}'}(\Omega), \quad \text{there exists} \quad (f_0, f_1, \ldots, f_N) \in \prod_{i=0}^{N} L^{p_i}(\Omega)$$
$$\text{such that} \quad F = f_0 - \sum_{i=1}^{N} D^i f_i.$$

It can be easily seen that $W^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ are separable and reflexive Banach spaces.

We will use later the following Sobolev embedding.

Lemma 2.1 *Let Ω be a bounded open set in $I\!R^N$. Then the following embeddings are compact.*

(i) If $\underline{p} < N$, then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \underline{p}^*[$, where $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N}$.

(ii) If $\underline{p} = N$, then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, +\infty[$.

(iii) If $\underline{p} > N$, then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous and from the compact embeddings theorem in classical Sobolev spaces. For more details, we refer the reader to [10,22].

The anisotropic parabolic space $L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega))$ is given by the formula

$$L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega)) = \left\{ u \text{ measurable function } \Big/ \sum_{i=0}^{N} \int_0^T \|D^i u\|_{L^{p_i}(\Omega)}^{p_i} \, dt < \infty \right\}, \tag{2.2}$$

endowed with the norm

$$\|u\|_{L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))} := \sum_{i=0}^{N} \|D^\alpha u\|_{L^{p_i}(Q_T)}.$$

We introduce the functional space $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ by

$$L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)) = \left\{ u \in L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega)) \Big/ u = 0 \text{ on } \partial\Omega \times [0, T] \right\}. \tag{2.3}$$

$L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega))$ and $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ are separable and reflexive Banach spaces.

Definition 2.1 The dual space of $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ is defined as follows

$$L^{\vec{p}'}(0, T; W^{-1,\vec{p}'}(\Omega)) = \left\{ F = f_0 - \sum_{i=1}^{N} D^i f_i, \text{ with } f_0 \in L^{p_0'}(Q_T), \ f_i \in L^{p_i'}(Q_T) \right\}. \tag{2.4}$$

We define a norm on the dual space by

$$\|F\|_{L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))} = \inf \left\{ \sum_{i=0}^{N} \|f_i\|_{L^{p_i'}(Q_T)} \text{ with } F = f_0 - \sum_{i=1}^{N} D^i f_i \right. \\ \left. \text{such that } f_0 \in L^{p_0'}(Q_T) \text{ and } f_i \in L^{p_i'}(Q_T) \right\}.$$

The duality of the spaces $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ is given by relation

$$\int_0^T \langle F, v\rangle\, dt = \sum_{i=0}^{N} \int_{Q_T} f_i D^i v\, dx \qquad \text{for all} \quad v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)).$$

Lemma 2.2 (cf. [21]) *Let B_0, B and B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Let us set*

$$Y = \big\{u \,:\, u \in L^{q_0}(0,T;B_0) \quad \text{and} \quad u' \in L^{q_1}(0,T;B_1)\big\}$$

where $q_0 > 1$ and $q_1 > 1$ are real numbers.

Assume that the embedding $B_0 \hookrightarrow\hookrightarrow B$ is compact. Then the embedding

$$Y \hookrightarrow\hookrightarrow L^{q_0}(0,T;B)$$

is continuous and compact.

Remark 2.1 Let $\underline{p} > \dfrac{2N}{N+2}$, we set

$$B_0 = W_0^{1,\vec{p}}(\Omega), \qquad B = L^2(\Omega) \qquad \text{and} \qquad B_1 = W^{-1,\vec{p}'}(\Omega),$$

then $W_0^{1,\vec{p}}(\Omega) \subset L^2(\Omega) \subset W^{-1,\vec{p}'}(\Omega)$ and the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ is compact.

We set $q_0 = \underline{p}$ and $q_1 = \min\{p_0', p_1', \ldots, p_N'\}$. In view of Lemma 2.2, we obtain

$$\big\{u \,:\, u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad u' \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\big\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.5}$$

Moreover, in view of [4] we have

$$\big\{u \,:\, u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad u' \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\big\} \subseteq C([0,T];L^1(\Omega)). \tag{2.6}$$

For more details, we refer the reader to [19].

3 Some technical Lemmas

Definition 3.1 Let $k > 0$, the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$ is defined as

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Lemma 3.1 (cf. [11], *Theorem 13.47)* *Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that* $u_n \to u$ *a.e. in* Ω, $u_n,\ u \geq 0$ *a.e. and* $\int_\Omega u_n\,dx \to \int_\Omega u\,dx$, *then* $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 3.2 *Let* $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *then* $T_k(u) \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *for any* $k > 0$. *Moreover, we have*

$$T_k(u) \longrightarrow u \quad in \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \qquad as \quad k \to \infty.$$

Proposition 3.1 *We introduce a time mollification of a function* $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *for all* $\mu \geq 0$ *by*

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \overline{u}(x,s) exp(\mu(s-t))\,ds \qquad where \quad \overline{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Then, the following assertions hold.

(i) If $u \in L^{p_0}(Q_T)$, *then* u_μ *is measurable in* Q_T, $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ *and*

$$\int_{Q_T} |u_\mu|^{p_0}\,dx\,dt \leq \int_{Q_T} |u|^{p_0}\,dx\,dt.$$

(ii) If $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, *then* $u_\mu \to u$ *in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *as* $\mu \to +\infty$.

(iii) If $u_n \to u$ *in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, *then* $(u_n)_\mu \to u_\mu$ *in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

(iv) $|(T_k(u))_\mu| \leq k$ *for all* $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

Proofs of Lemma 3.2 and Proposition 3.1 are similar to those in the classical space $L^p(0,T;W_0^{1,p}(\Omega))$.

Lemma 3.3 *Assume (1.2)–(1.4) hold. Let* $(u_n)_n$ *be a sequence in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *such that* $\dfrac{\partial u_n}{\partial t} \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$, $u_n \rightharpoonup u$ *in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *and*

$$\begin{aligned}\sum_{i=1}^{N} \int_{Q_T} \big(a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u)\big)(D^i u_n - D^i u)\,dx\,dt \\ + \int_{Q_T} \Big(|u_n|^{p_0-2}u_n - |u|^{p_0-2}u\Big)(u_n - u)\,dx\,dt \longrightarrow 0,\end{aligned} \tag{3.1}$$

then $u_n \to u$ *in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *for a subsequence.*

Proof Let

$$D_n(x,t) = \sum_{i=1}^{N} \big(a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u)\big)(D^i u_n - D^i u)$$

$$+\big(|u_n|^{p_0-2}u_n - |u|^{p_0-2}u\big)(u_n - u).$$

Thanks to (1.4) we have D_n is a positive function, and in view of (3.1), we get $D_n \to 0$ in $L^1(Q_T)$ *as* $n \to \infty$.

On one hand, since $u_n \rightharpoonup u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, and in view of the compact embedding (2.5), we have $u_n \to u$ strongly in $L^2(Q_T)$, so that $u_n \to u$ a.e in Q_T, and since $D_n \to 0$ a.e in Q_T, there exists a subset B in Q_T with measure zero such that for any $(x,t) \in Q_T \backslash B$

$$|u_n(x,t)| < \infty, \quad |D^i u_n(x,t)| < \infty, \quad |K_i(x,t)| < \infty,$$
$$u_n(x,t) \to u(x,t) \quad \text{and} \quad D_n(x,t) \to 0.$$

On the other hand, we have

$$\begin{aligned} D_n(x,t) &= \sum_{i=1}^{N}\big(a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u)\big)(D^i u_n - D^i u) \\ &\quad +\big(|u_n|^{p_0-2}u_n - |u|^{p_0-2}u\big)(u_n - u) \\ &\geq \alpha \sum_{i=1}^{N} |D^i u_n|^{p_i} \\ &\quad +\alpha \sum_{i=1}^{N} |D^i u|^{p_i} + |u_n|^{p_0} + |u|^{p_0} - \beta \sum_{i=1}^{N}\big(K(x,t) + |D^i u_n|^{p_i-1}\big)|D^i u| \\ &\qquad -\beta \sum_{i=1}^{N}\big(K(x,t) + |D^i u|^{p_i-1}\big)|D^i u_n| - |u_n|^{p_0-1}|u| - |u|^{p_0-1}|u_n| \\ &\geq \underline{\alpha} \sum_{i=0}^{N} |D^i u_n|^{p_i} - C_{x,t} \sum_{i=0}^{N}\Big(1 + |D^i u_n|^{p_i-1} + |D^i u_n|\Big), \end{aligned}$$

where $\underline{\alpha} = \min(1,\alpha)$ and $C_{x,t}$ is a constant depending on (x,t), without dependence on n. It follows that

$$D_n(x,t) \geq \sum_{i=0}^{N} |D^i u_n|^{p_i}\Big(\underline{\alpha} - \frac{C_{x,t}}{|D^i u_n|^{p_i}} - \frac{C_{x,t}}{|D^i u_n|} - \frac{C_{x,t}}{|D^i u_n|^{p_i-1}}\Big).$$

By a standard argument $(D^i u_n)_n$ is bounded almost everywhere in Q_T, (Indeed, if $|D^i u_n| \to \infty$ in a measurable subset $E \subset Q_T$, then

$$\lim_{n\to\infty}\int_{Q_T} D_n(x,t)\,dx\,dt \geq \lim_{n\to\infty}\sum_{i=0}^{N}$$
$$\int_E |D^i u_n|^{p_i}\big(\underline{\alpha} - \frac{C_{x,t}}{|D^i u_n|^{p_i}} - \frac{C_{x,t}}{|D^i u_n|} - \frac{C_{x,t}}{|D^i u_n|^{p_i-1}}\big)\,dx\,dt = \infty,$$

which is absurd since $D_n \to 0$ in $L^1(Q_T)$).

Let ξ_i^* be an accumulation point of $(D^i u_n)_n$, we have $|\xi_i^*| < \infty$. By (3.1) and the continuity of the Carathéodory function $a(x,t,\cdot)$, we obtain

$$\Big(a_i(x,t,\xi^*) - a_i(x,t,\nabla u)\Big)(\xi_i^* - D^i u) = 0.$$

Thanks to (1.4), we get $\xi_i^* = D^i u$, and the uniqueness of the accumulation point implies that $D^i u_n(x,t) \to D^i u(x,t)$ a.e in Q_T for $i = 0, 1, \cdots, N$.

Now since $(a_i(x,t,\nabla u_n))_n$ is bounded in $L^{p_i'}(Q_T)$ and $a_i(x,t,\nabla u_n) \to a_i(x,t,\nabla u)$ a.e in Q_T, one can establish that

$$a_i(x,t,\nabla u_n) \rightharpoonup a_i(x,t,\nabla u) \quad in \quad L^{p_i'}(Q_T).$$

Using (3.1) and Lemma 3.1, we deduce that

$$|u_n|^{p_i} \longrightarrow |u|^{p_i} \quad \text{in} \quad L^1(Q_T), \tag{3.2}$$

and

$$a_i(x,t,\nabla u_n) D^i u_n \longrightarrow a_i(x,t,\nabla u) D^i u \quad \text{in} \quad L^1(Q_T). \tag{3.3}$$

According to the condition (1.3), we have

$$\alpha |D^i u_n|^{p_i} \le a_i(x,t,\nabla u_n) D^i u_n \quad \text{for} \quad i = 1, \ldots, N.$$

Let $y_n^i = \dfrac{a_i(x,t,\nabla u_n) D^i u_n}{\alpha}$ and $y^i = \frac{a_i(x,t,\nabla u) D^i u}{\alpha}$. In view of Fatou's lemma, we get

$$\int_{Q_T} 2y^i \, dx\, dt \le \liminf_{n\to\infty} \int_{Q_T} (y_n^i + y^i - \frac{1}{2^{p_i - 1}} |D^i u_n - D^i u|^{p_i}) \, dx\, dt.$$

Hence $0 \le -\limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt$, and since

$$0 \le \liminf_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt \le \limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt \le 0,$$

it follows that $\int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt \longrightarrow 0$ as $n \to \infty$. Then we obtain

$$D^i u_n \longrightarrow D^i u \quad \text{in} \quad L^{p_i}(Q_T) \quad \text{for} \quad i = 1, \ldots, N.$$

Therefore, thanks to (3.2), we deduce that

$$u_n \longrightarrow u \qquad \text{in} \quad L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega)).$$

This completes our proof. □

4 Main results

For all $k > 0$ and $s \in I\!R$, we define

$$\varphi_k(r) = \int_0^r T_k(s)\, ds = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \le k, \\ k|r| - \frac{k^2}{2} & \text{if} \quad |r| > k. \end{cases}$$

Definition 4.1 A measurable function u is an entropy solution of the parabolic problem (1.1) if

$$T_k(u) \in L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)), \qquad g(x, t, u, \nabla u) \in L^1(Q_T), \qquad |u|^{p_0-2}u \in L^1(Q_T),$$

and

$$\begin{aligned} &\int_\Omega \varphi_k(u-\psi)(T)\, dx - \int_\Omega \varphi_k(u-\psi)(0)\, dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\, dx\, dt \\ &+ \sum_{i=1}^N \int_{Q_T} a_i(x, t, \nabla u)\; D^i T_k(u-\psi)\, dx\, dt + \int_{Q_T} g(x, t, u, \nabla u) T_k(u-\psi)\, dx\, dt \\ &\quad + \int_{Q_T} d(x, t)|u|^{p_0-2}u\; T_k(u-\psi)\, dx\, dt \\ &\quad \le \int_{Q_T} f T_k(u-\psi)\, dx\, dt + \sum_{i=1}^N \int_{Q_T} \phi_i(u)\; D^i T_k(u-\psi)\, dx\, dt, \end{aligned} \tag{4.1}$$

for all $\psi \in L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$ with $\dfrac{\partial \psi}{\partial t} \in L^{\vec{p}'}(0, T; W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$.

Theorem 4.1 *Assume (1.2)–(1.6) hold, with* $f \in L^1(Q_T)$, $u_0 \in L^1(Q_T)$ *and* $\phi(\cdot) \in C^0(I\!R, I\!R^N)$. *Then the problem (1.1) has at least one entropy solution.*

Proof.

Step 1: Approximate problem. Let $(f_n)_n$ be a sequence in $L^{\vec{p}'}(0, T; W^{-1,\vec{p}'}(\Omega)) \cap L^1(Q_T)$ such that $f_n \longrightarrow f$ in $L^1(Q_T)$ with $|f_n| \le |f|$, and let $(u_{0,n})_n$ be a sequence in $\mathcal{C}_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $|u_{0,n}| \le |u_0|$. Consider the approximate problem

$$\begin{cases} (u_n)_t + Au_n + g_n(x, t, u_n, \nabla u_n) = f_n - \operatorname{div} \phi_n(u_n) & \text{in} \quad Q_T, \\ u_n(x, t) = 0 & \text{on} \quad \Sigma_T, \\ u_n(x, 0) = u_{0,n} & \text{in} \quad \Omega, \end{cases} \tag{4.2}$$

where $\phi_n(s) = (\phi_{1,n}(s), \ldots, \phi_{N,n}(s))$ with $\phi_{i,n}(s) = \phi_i(T_n(s))$, $i = 1, \ldots, N$, and the nonlinear term $g_n(x, t, s, \xi) = T_n(g(x, t, s, \xi))$. Note that

$$g_n(x,t,s,\xi)s \geq 0, \qquad |g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)| \qquad \textit{and}$$
$$|g_n(x,t,s,\xi)| \leq n \qquad \forall n \in I\!N^*.$$

We define the operator $G_n : L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega)) \longmapsto L^{\vec{p}'}(0,T; W^{-1,\vec{p}'}(\Omega))$ by

$$\int_0^T \langle G_n u, v\rangle\, dt = \int_{Q_T} g_n(x,t,u,\nabla u) v\, dx\, dt$$
$$-\sum_{i=1}^N \int_{Q_T} \phi_{i,n}(u) D^i v\, dx\, dt \qquad \forall v \in L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega)).$$

Thanks to Hölder's inequality, we have

$$\begin{aligned}\Big|\int_0^T \langle G_n u, v\rangle\, dt\Big| &\leq \Big(\frac{1}{p_0} + \frac{1}{p_0'}\Big)\, \|g_n(x,t,u,\nabla u)\|_{L^{p_0'}(Q_T)} \|v\|_{L^{p_0}(Q_T)} \\ &\quad + \sum_{i=1}^N \Big(\frac{1}{p_i} + \frac{1}{p_i'}\Big) \|\phi_n(u)\|_{L^{p_i'}(Q_T)} \|D^i v\|_{L^{p_i}(Q_T)} \\ &\leq 2\Big(n \,\mathrm{meas}(Q_T)^{\frac{1}{p_0'}} + \sum_{i=1}^N \sup_{|s|\leq n} |\phi_i(s)|\, \mathrm{meas}(Q_T)^{\frac{1}{p_i'}}\Big) \|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \\ &\leq C_0 \|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}\end{aligned} \tag{4.3}$$

for all $u,\ v \in L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$.

In view of Lemma 5.1 (see Appendix), the operator $B_n = A + G_n$ is bounded, pseudo-monotone and coercive. Using (2.6), we conclude that there exists at least one weak solution $u_n \in L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ of problem (4.2) (cf. [14]).

Step 2: Weak convergence of truncations. Let $k \geq 1$. By taking $T_k(u_n)$ as a test function in (4.2), we obtain

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)\rangle\, dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u_n)\, D^i T_k(u_n)\, dx\, dt \\ &+ \int_{Q_T} d(x,t)|u_n|^{p_0-2} u_n T_k(u_n)\, dx\, dt \\ &+ \int_{Q_T} g_n(x,t,u_n,\nabla u_n) T_k(u_n)\, dx\, dt \\ &= \int_{Q_T} f_n T_k(u_n)\, dx\, dt + \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(u_n)\, D^i T_k(u_n)\, dx\, dt.\end{aligned} \tag{4.4}$$

We have $\varphi_k(r) = \int_0^r T_k(s)\,ds$ and since $|\varphi_k(r)| \le k|r|$, then

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)\rangle\,dt = \int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_k(u_n)\,dt\,dx = \int_\Omega \int_0^T \frac{\partial \varphi_k(u_n)}{\partial t}\,dt\,dx$$
$$= \int_\Omega \varphi_k(u_n(T))\,dx - \int_\Omega \varphi_k(u_{0,n})\,dx$$
$$\ge \int_\Omega \varphi_k(u_n(T))\,dx - k\|u_0\|_{L^1(\Omega)}. \tag{4.5}$$

For the second and third terms on the left-hand side of (4.4), we have

$$\int_{Q_T} a_i(x,t,\nabla u_n)\,D^i T_k(u_n)\,dx\,dt \ge \alpha \int_{Q_T} |D^i T_k(u_n)|^{p_i}\,dx\,dt \tag{4.6}$$

and

$$\int_{Q_T} d(x,t)|u_n|^{p_0-2}u_n T_k(u_n)\,dx\,dt \ge d_0 \int_{Q_T} |T_k(u_n)|^{p_0}\,dx\,dt. \tag{4.7}$$

Having in mind the sign condition, the fourth term on the left-hand side of (4.4) is positive. Concerning the two terms on the right-hand side of (4.4),we have

$$\int_{Q_T} f_n\,T_k(u_n)\,dx\,dt \le k \int_{Q_T} |f_n|\,dx\,dt \le k\|f\|_{L^1(Q_T)}. \tag{4.8}$$

Taking $\Phi_{i,n}(s) = \int_0^s \phi_{i,n}(\sigma)\,d\sigma$, then $\Phi_{i,n}(0) = 0$ and $\Phi_{i,n}(\cdot) \in C^1(I\!R)$, and in view of Green formula, we obtain

$$\int_{Q_T} \phi_{i,n}(u_n) D^i T_k(u_n)\,dx\,dt$$
$$= \int_0^T \int_\Omega D^i \Phi_{i,n}(T_k(u_n))\,dx\,dt$$
$$= \int_0^\tau \int_{\partial\Omega} \Phi_{i,n}(T_k(u_n)).n_i\,d\sigma\,dt = 0, \tag{4.9}$$

since $u_n = 0$ on $\partial\Omega \times (0,T)$, where $\vec{n} = (n_1, n_2, \ldots, n_N)$ is the exterior normal vector on the boundary $\partial\Omega \times (0,T)$.

By combining (4.4)–(4.9), we deduce that

$$\int_\Omega \varphi_k(u_n(T))\,dx + \alpha \sum_{i=1}^N \int_{Q_T} |D^i T_k(u_n)|^{p_i}\,dx\,dt$$
$$+d_0 \int_{Q_T} |T_k(u_n)|^{p_0}\,dx\,dt \le k(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q_T)}). \tag{4.10}$$

Since $\varphi_k(u_n(T)) \geq 0$, then there exists a constant C_2 that does not depend on n and k, such that

$$\begin{aligned}
&\sum_{i=1}^{N} \|D^i T_k(u_n)\|_{L^{p_i}(Q_T)}^{\underline{p}} + \|T_k(u_n)\|_{L^{p_0}(Q_T)}^{\underline{p}} \\
&\leq \sum_{i=1}^{N} \int_{Q_T} |D^i T_k(u_n)|^{p_i}\, dx\, dt + \int_{Q_T} |T_k(u_n)|^{p_0}\, dx\, dt + N + 1 \\
&\leq kC_3.
\end{aligned}$$

Thus, we get

$$\|T_k(u_n)\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \leq C_3 k^{\frac{1}{\underline{p}}}. \tag{4.11}$$

Let $k \geq 1$, we have

$$\begin{aligned}
&k \text{ meas}\{|u_n| > k\} \\
&= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx\, dt \leq \int_{Q_T} |T_k(u_n)|\, dx\, dt \\
&\leq meas(Q_T)^{\frac{1}{p_0'}} \|T_k(u_n)\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \\
&\leq C_4 k^{\frac{1}{\underline{p}}},
\end{aligned}$$

which implies that

$$\text{meas}\{|u_n| > k\} \leq C_4 \frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{4.12}$$

For all $\lambda > 0$, we have

$$\begin{aligned}
\text{meas}\{|u_n - u_m| > \lambda\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\
&+ \text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\}.
\end{aligned} \tag{4.13}$$

On one hand, using (4.12), we get for all $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0(\varepsilon). \tag{4.14}$$

On the other hand, in view of (4.11), $(T_k(u_n))_n$ is bounded in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$. Then there exists a sequence still denoted by $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{as} \quad n \to +\infty,$$

and by the compact embedding (2.5), we obtain

$$T_k(u_n) \longrightarrow \eta_k \quad \text{in} \quad L^1(Q_T) \quad \text{and} \quad \text{a.e in} \quad Q_T.$$

Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q_T. Hence for all $k > 0$ and $\lambda,\ \varepsilon > 0$ there exists $n_0 = n_0(k, \lambda, \varepsilon)$ such that

$$\text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \lambda\big\} \leq \frac{\varepsilon}{3} \qquad \forall n, m \geq n_0. \tag{4.15}$$

By combining (4.13)–(4.15), we deduce that for all $\varepsilon, \lambda > 0$, there exists $n_0 = n_0(\lambda, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \lambda\} \leq \varepsilon \qquad \forall n, m \geq n_0. \tag{4.16}$$

If follows that $(u_n)_n$ is a Cauchy sequence in measure. So that there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n \longrightarrow u \qquad \text{a.e in} \quad Q_T.$$

Hence we have

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{in} \quad L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)). \tag{4.17}$$

Applying Lebesgue dominated convergence theorem, we get

$$T_k(u_n) \longrightarrow T_k(u) \qquad \text{in} \quad L^{p_0}(Q_T). \tag{4.18}$$

Step 3: A priori estimates. Let $h > 0$, taking $T_{h+1}(u_n) - T_h(u_n)$ as a test function in (4.2), we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_{h+1}(u_n) - T_h(u_n)\rangle \, dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} a_i(x, t, \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n)) \, dx \, dt \\
&\quad + \int_{Q_T} g_n(x, t, u_n, \nabla u_n)(T_{h+1}(u_n) - T_h(u_n)) \, dx \, dt \\
&\quad + \int_{Q_T} d(x, t)|u_n|^{p_0-2} u_n (T_{h+1}(u_n) - T_h(u_n)) \, dx \, dt \\
&\quad = \int_{Q_T} f_n \cdot (T_{h+1}(u_n) - T_h(u_n)) \, dx \, dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(u_n) \cdot (D^i T_{h+1}(u_n) - D^i T_h(u_n)) \, dx \, dt.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_{h+1}(u_n) - T_h(u_n)\rangle \, dt \\
&= \int_\Omega \int_0^T \frac{\partial \varphi_{h+1}(u_n)}{\partial t} \, dt \, dx - \int_\Omega \int_0^T \frac{\partial \varphi_h(u_n)}{\partial t} \, dt \, dx \\
&= \int_\Omega \varphi_{h+1}(u_n(T)) - \varphi_{h+1}(u_{0,n}) \, dx - \int_\Omega \varphi_h(u_n(T)) - \varphi_h(u_{0,n}) \, dx
\end{aligned}$$

with

$$\begin{aligned}
&\int_\Omega \varphi_{h+1}(u_n(T)) \, dx - \int_\Omega \varphi_h(u_n(T)) \, dx \\
&= \int_{\{h \le |u_n(T)| < h+1\}} \left(\frac{u_n^2(T)}{2} - h|u_n(T)| + \frac{h^2}{2}\right) dx \\
&+ \int_{\{h+1 \le |u_n(T)|\}} \left(|u_n(T)| - h - \frac{1}{2}\right) dx \ge 0. \qquad (4.19)
\end{aligned}$$

Similarly to (4.9), we have

$$\begin{aligned}
&\int_{Q_T} \phi_{i,n}(u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n)) \, dx \, dt \\
&= \int_{Q_T} \phi_{i,n}(T_{h+1}(u_n)) D^i T_{h+1}(u_n) \, dx \, dt \\
&\quad - \int_{Q_T} \phi_{i,n}(T_h(u_n)) D^i T_h(u_n) \, dx \, dt \\
&= \int_{Q_T} D^i \Phi_{i,n}(T_{h+1}(u_n)) \, dx \, dt \\
&\quad - \int_{Q_T} D^i \Phi_{i,n}(T_h(u_n)) \, dx \, dt = 0. \qquad (4.20)
\end{aligned}$$

Now since $T_{h+1}(u_n) - T_h(u_n)$ has the same sign as u_n and $|T_{h+1}(u_n) - T_h(u_n)| \le 1$, we get

$$\begin{aligned}
&\alpha \sum_{i=1}^N \int_{\{h \le |u_n| < h+1\}} |D^i u_n|^{p_i} \, dx \, dt + d_0 \int_{\{h+1 \le |u_n|\}} |u_n|^{p_0 - 1} \, dx \, dt \\
&\le \int_{\{|u_n| \ge h\}} |f| \, dx \, dt + \int_\Omega \varphi_{h+1}(u_{0,n}) \, dx - \int_\Omega \varphi_h(u_{0,n}) \, dx.
\end{aligned} \qquad (4.21)$$

Concerning the terms on the right-hand side of (4.21), we have

$$\int_{\{|u_n| \ge h\}} |f| \, dx \, dt \longrightarrow 0 \quad \text{as} \quad h \to \infty, \qquad (4.22)$$

and since $u_{0,n} \to u_0$ in $L^1(\Omega)$, then

$$\begin{aligned}
&\int_\Omega \varphi_{h+1}(u_{0,n})\,dx - \int_\Omega \varphi_h(u_{0,n})\,dx \\
&= \int_{\{h\le|u_{0,n}|<h+1\}} (\frac{|u_{0,n}|^2}{2} - h|u_{0,n}| + \frac{h^2}{2})\,dx \\
&+ \int_{\{h+1\le|u_{0,n}|\}} (|u_{0,n}| - h - \frac{1}{2})\,dx \\
&\le \int_{\{h\le|u_{0,n}|<h+1\}} \frac{1}{2}\,dx \\
&+ \int_{\{h+1\le|u_{0,n}|\}} |u_0|\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty.
\end{aligned} \tag{4.23}$$

By combining (4.21)–(4.23), we deduce that

$$\sum_{i=1}^N \int_{\{h\le|u_n|<h+1\}} |D^i u_n|^{p_i}\,dx\,dt \to 0 \quad \text{as} \quad h\to\infty, \tag{4.24}$$

and

$$\int_{\{h+1\le|u_n|\}} |u_n|^{p_0-1}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{4.25}$$

Step 4: Convergence of the gradient. In the sequel, we denote by $\varepsilon_i(n),\ i=1,\ 2,\dots$ various functions of real numbers which converges to 0 as n tends to infinity (respectively for $\varepsilon_i(n,\mu)$ and $\varepsilon_i(n,\mu,h)$).

Let $\xi_k(s) = s.\exp(\gamma s^2)$ where $\gamma = \left(\frac{b(k)}{2\alpha}\right)^2$. It is obvious that

$$\xi_k'(s) - \frac{b(k)}{\alpha}|\xi_k(s)| \ge \frac{1}{2} \qquad \forall s\in I\!R.$$

Let $\omega_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$, where $(T_k(u))_\mu$ is the mollification, with respect to time, of $T_k(u)$.

Let $h\ge k>0$, taking $S_h(\cdot)\in C^2(I\!R)$ an increasing function, such that $S_h(r)=r$ for $|r|\le h$ and $\text{supp}(S_h')\subset[-h-1,h+1]$, then $\text{supp}(S_h'')\subset[-h-1,-h]\cup[h,h+1]$.

Since $T_k(u_n)-(T_k(u))_\mu$ have the same sign as u_n on the set $\{|u_n|>k\}$, then, by using $\xi_k(\omega_{n,\mu})S_h'(u_n)$ as a test function in (4.2), we obtain

$$\mathcal{J}^1_{n,\mu,h} + \mathcal{J}^2_{n,\mu,h} + \mathcal{J}^3_{n,\mu,h} + \mathcal{J}^4_{n,\mu,h} + \mathcal{J}^5_{n,\mu,h} \le \mathcal{J}^6_{n,\mu,h} + \mathcal{J}^7_{n,\mu,h} + \mathcal{J}^8_{n,\mu,h}, \tag{4.26}$$

where

$$
\begin{aligned}
\mathcal{J}^1_{n,\mu,h} &= \int_0^T \langle \frac{\partial u_n}{\partial t}, \xi_k(\omega_{n,\mu}) S'_h(u_n)\rangle \, dt,\\
\mathcal{J}^2_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} S'_h(u_n) a_i(x,t,\nabla u_n)(D^i T_k(u_n) - D^i (T_k(u))_\mu) \xi'_k(\omega_{n,\mu})\, dx\, dt,\\
\mathcal{J}^3_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} \xi_k(\omega_{n,\mu}) S''_h(u_n) a_i(x,t,\nabla u_n)\, D^i u_n \, dx\, dt,\\
\mathcal{J}^4_{n,\mu,h} &= \int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\xi_k(\omega_{n,\mu})\, dx\, dt,\\
\mathcal{J}^5_{n,\mu,h} &= \int_{\{|u_n|\le k\}} d(x,t)|u_n|^{p_0-2} u_n \xi_k(\omega_{n,\mu})\, dx\, dt,\\
\mathcal{J}^6_{n,\mu,h} &= \int_{Q_T} f_n S'_h(u_n)\xi_k(\omega_{n,\mu})\, dx\, dt,\\
\mathcal{J}^7_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} S'_h(u_n)\phi_{i,n}(u_n)(D^i T_k(u_n) - D^i (T_k(u))_\mu)\xi'_k(\omega_{n,\mu})\, dx\, dt,\\
\mathcal{J}^8_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(u_n) D^i u_n S''_h(u_n)\xi_k(\omega_{n,\mu})\, dx\, dt.
\end{aligned}
\tag{4.27}
$$

<u>**The first term:**</u> We have

$$
\begin{aligned}
\mathcal{J}^1_{n,\mu,h} &= \int_{Q_T} \frac{\partial S_h(u_n)}{\partial t} \xi_k(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt\\
&= \int_{Q_T} \frac{\partial (S_h(u_n) - T_k(u_n))}{\partial t} \xi_k(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt\\
&\quad + \int_{Q_T} \frac{\partial T_k(u_n)}{\partial t} \xi_k(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt\\
&= \Big[\int_\Omega (S_h(u_n) - T_k(u_n))\xi_k(T_k(u_n) - (T_k(u))_\mu)\, dx \Big]_0^T\\
&\quad - \int_{Q_T} (S_h(u_n) - T_k(u_n))\xi'_k(T_k(u_n) - (T_k(u))_\mu)\Big(\frac{\partial T_k(u_n)}{\partial t} - \frac{\partial (T_k(u))_\mu}{\partial t}\Big)\, dx\, dt\\
&\quad + \int_{Q_T} \frac{\partial T_k(u_n)}{\partial t} \xi_k(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt\\
&= I_1 + I_2 + I_3.
\end{aligned}
\tag{4.28}
$$

Concerning the first term on the right hand side of (4.28), we have $S_h(u_n) = T_k(u_n) = u_n$ on $\{|u_n| \le k\}$, and $|S_h(u_n)| \ge |T_k(u_n)|$ on the set $\{|u_n| > k\}$. Since $S_h(u_n)$ and $T_k(u_n)$ have the same sign of u_n, we obtain

$$
\begin{aligned}
I_1 &= \Big[\int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k(T_k(u_n) - (T_k(u))_\mu)\, dx \Big]_0^T\\
&\ge - \int_{\{|u_{0,n}|>k\}} (S_h(u_{0,n}) - T_k(u_{0,n}))\xi_k(T_k(u_{0,n}) - (T_k(u_0))_\mu)\, dx.
\end{aligned}
$$

Since $(T_k(u_0))_\mu = T_k(u_0)$, we deduce that $I_1 \ge \varepsilon_1(n)$, with

$$\varepsilon_1(n) = -\int_{\{|u_{0,n}|>k\}} (S_h(u_{0,n}) - T_k(u_{0,n}))\xi_k(T_k(u_{0,n}) - T_k(u_0))\,dx \longrightarrow$$
0 as $n \to \infty$.

For the second term on the right-hand side of (4.28), we have $(S_h(u_n) - T_k(u_n))\dfrac{\partial T_k(u_n)}{\partial t} = 0$. Hence

$$\begin{aligned}
I_2 &= \int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)\frac{\partial (T_k(u))_\mu}{\partial t}\,dx\,dt \\
&= \mu\int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u) - (T_k(u))_\mu)\,dx\,dt \\
&= \mu\int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u) - T_k(u_n))\,dx\,dt \\
&\quad +\mu\int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\ge \mu\int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u) - T_k(u_n))\,dx\,dt.
\end{aligned}$$

It follows that $I_2 \ge \varepsilon_2(n)$.

Concerning the last term I_3, let $\Psi(s) = \dfrac{1}{2\gamma}\exp(\gamma s^2)$, then $\Psi'(s) = \xi_k(s)$, and we obtain

$$\begin{aligned}
I_3 &= \int_{Q_T} \frac{\partial (T_k(u_n) - (T_k(u))_\mu)}{\partial t}\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\quad + \int_{Q_T} \frac{\partial (T_k(u))_\mu}{\partial t}\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&= \Big[\int_\Omega \Psi(T_k(u_n) - (T_k(u))_\mu)\,dx\Big]_0^T \\
&\quad +\mu\int_{Q_T} (T_k(u) - (T_k(u))_\mu)\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\ge -\int_\Omega \Psi(T_k(u_{0,n}) - T_k(u_0))\,dx \\
&\quad +\mu\int_{Q_T} (T_k(u) - (T_k(u))_\mu)\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\ge \varepsilon_3(n) \\
&\quad +\mu\int_{Q_T} (T_k(u) - (T_k(u))_\mu)\xi_k(T_k(u) - (T_k(u))_\mu)\,dx\,dt \\
&\ge \varepsilon_3(n).
\end{aligned}$$

Combining these last estimates, we conclude that

$$\mathcal{J}^1_{n,\mu,h} \ge \varepsilon_4(n), \tag{4.29}$$

with $\varepsilon_4(n) = \varepsilon_1(n) + \varepsilon_2(n) + \varepsilon_3(n)$.

<u>The second term:</u> We have $S'_h(s) \geq 0$ and $S'_h(s) = 1$ for $|s| \leq k$, with $\operatorname{supp}(S'_h) \subset [-h-1, h+1]$, then

$$
\begin{aligned}
\mathcal{J}^2_{n,\mu,h} = & \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} a_i(x,t,\nabla T_k(u_n))(D^i T_k(u_n) - D^i (T_k(u))_\mu)\xi'_k(\omega_{n,\mu})\, dx\, dt \\
& - \sum_{i=1}^{N} \int_{\{k<|u_n| \leq h+1\}} S'_h(u_n) a_i(x,t,\nabla T_{h+1}(u_n))\ D^i (T_k(u))_\mu \xi'_k(\omega_{n,\mu})\, dx\, dt \\
\geq & \sum_{i=1}^{N} \int_{Q_T} \Big(a_i(x,t,\nabla T_k(u_n)) \\
& - a_i(x,t,\nabla T_k(u)) \Big) (D^i T_k(u_n) - D^i T_k(u))\xi'_k(\omega_{n,\mu})\, dx\, dt \\
& - \xi'_k(2k) \sum_{i=1}^{N} \int_{Q_T} |a_i(x,t,\nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)|\, dx\, dt \\
& - \xi'_k(2k) \sum_{i=1}^{N} \int_{\{|u_n|>k\}} |a_i(x,t,\nabla T_k(u_n))|\, |D^i (T_k(u))_\mu|\, dx\, dt \\
& - \xi'_k(2k) \sum_{i=1}^{N} \int_{Q_T} |a_i(x,t,\nabla T_k(u_n))|\, |D^i T_k(u) - D^i (T_k(u))_\mu|\, dx\, dt \\
& - \xi'_k(2k) \|S'_h\|_{L^\infty(I\!R)} \\
& \sum_{i=1}^{N} \int_{\{k<|u_n| \leq h+1\}} |a_i(x,t,\nabla T_{h+1}(u_n))| |D^i (T_k(u))_\mu|\, dx\, dt.
\end{aligned} \tag{4.30}
$$

Since $a_i(x,t,\nabla T_k(u))$ is bounded in $L^{p'_i}(Q_T)$, and $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ in $L^{p_i}(Q_T)$, we get

$$
\int_{Q_T} |a_i(x,t,\nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{4.31}
$$

For the three last terms on the right-hand side of (4.30), we have $|a_i(x,t,\nabla T_k(u_n))|$ is bounded in $L^{p'_i}(Q_T)$, then there exists $\vartheta_i \in L^{p'_i}(Q_T)$ such that $|a_i(x,t,\nabla T_k(u_n))| \rightharpoonup \vartheta_i$ weakly in $L^{p'_i}(Q_T)$, and since $D^i (T_k(u))_\mu \to D^i T_k(u)$ in $L^{p_i}(Q_T)$, it follows that

$$
\int_{\{|u_n|>k\}} |a_i(x,t,\nabla T_k(u_n))|\, |D^i (T_k(u))_\mu|\, dx\, dt \longrightarrow \int_{\{|u|>k\}} \vartheta_i\, |D^i T_k(u)|\, dx\, dt = 0 \quad \text{as} \quad \mu,\ n \to \infty. \tag{4.32}
$$

Similarly, we can prove that

$$
\int_{Q_T} |a_i(x,t,\nabla T_k(u_n))|\, |D^i T_k(u) - D^i (T_k(u))_\mu|\, dx\, dt \longrightarrow 0 \text{ as } \mu \text{ and } n \to \infty, \tag{4.33}
$$

and

$$\int_{\{k<|u_n|\leq h+1\}} |a_i(x,t,\nabla T_{h+1}(u_n))|\,|D^i(T_k(u))_\mu|\,dx\,dt \longrightarrow 0 \text{ as } \mu \text{ and } n\to\infty. \tag{4.34}$$

By combining (4.30)–(4.34), we deduce that

$$\begin{aligned}&\mathcal{J}^2_{n,\mu,h}\\ &\geq \sum_{i=1}^N \int_{Q_T} \Big(a_i(x,t,\nabla T_k(u_n)) - a_i(x,t,\nabla T_k(u))\Big)(D^i T_k(u_n)\\ &\quad - D^i T_k(u))\xi_k'(\omega_{n,\mu})\,dx\,dt + \varepsilon_5(\mu,n).\end{aligned} \tag{4.35}$$

<u>The third term:</u> We have $\operatorname{supp}(S_h'') \subset [-h-1,-h]\cup[h,h+1]$, and in view of Young's inequality, we obtain

$$\begin{aligned}\big|\mathcal{J}^3_{n,\mu,h}\big| &\leq \|S_h''\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} |a_i(x,t,\nabla T_{h+1}(u_n))|\,|\xi_k(\omega_{n,\mu})|\,|D^i T_{h+1}(u_n)|\,dx\,dt\\ &\leq \beta\|S_h''\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} (K_i(x,t)+|D^i T_{h+1}(u_n)|^{p_i-1})|\xi_k(\omega_{n,\mu})|\,|D^i T_{h+1}(u_n)|\,dx\,dt\\ &\leq \beta\|S_h''\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} |\xi_k(\omega_{n,\mu})|\,\frac{|K_i(x,t)|^{p_i'}}{p_i'}\,dx\,dt\\ &\quad+\beta|\xi_k(2k)|\,\|S_h''\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} (\frac{1}{p_i}+1)|D^i T_{h+1}(u_n)|^{p_i}\,dx\,dt.\end{aligned}$$

Since $\omega_{n,\mu} = T_k(u_n) - (T_k(u))_\mu \rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, then

$$\int_{\{h<|u_n|\leq h+1\}} |\xi_k(\omega_{n,\mu})|\,\frac{|K_i(x,t)|^{p_i'}}{p_i'}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad \mu,n\to\infty,$$

Thanks to (4.24), we obtain

$$\int_{\{h<|u_n|\leq h+1\}} (\frac{1}{p_i}+1)|D^i T_{h+1}(u_n)|^{p_i}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h\to\infty.$$

It follows that

$$\mathcal{J}^3_{n,\mu,h} \longrightarrow 0 \quad \text{as} \quad \mu,\ n \text{ then } h\to\infty. \tag{4.36}$$

<u>The fourth term:</u> Using (1.3) and (1.6), we have

$$
\begin{aligned}
\left|\mathcal{J}^4_{n,\mu,h}\right| &\leq \int_{\{|u_n|\leq k\}} |g_n(x,t,T_k(u_n),\nabla T_k(u_n))||\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&\leq b(k)\int_{\{|u_n|\leq k\}} (c(x,t)+\sum_{i=1}^N |D^i T_k(u_n)|^{p_i})|\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&\leq b(k)\int_{\{|u_n|\leq k\}} c(x,t)|\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_{Q_T}(a_i(x,t,\nabla T_k(u_n)) \\
&-a_i(x,t,\nabla T_k(u)))(D^i T_k(u_n)-D^i T_k(u))\,|\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_{Q_T} a_i(x,t,\nabla T_k(u))(D^i T_k(u_n)-D^i T_k(u))\,|\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_{Q_T} a_i(x,t,\nabla T_k(u_n))\,D^i T_k(u)\,|\xi_k(\omega_{n,\mu})|\,dx\,dt. \qquad (4.37)
\end{aligned}
$$

We also have $\xi_k(T_k(u_n)-(T_k(u))_\mu)\rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$. This implies that

$$
\int_{\{|u_n|\leq k\}} c(x,t)|\xi_k(\omega_{n,\mu})|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,\mu\to\infty. \qquad (4.38)
$$

Concerning the third and last terms on the right-hand side of (4.37), we have

$$
\begin{aligned}
&\left|\int_{Q_T} a_i(x,t,\nabla T_k(u))(D^i T_k(u_n)-D^i T_k(u))|\xi_k(\omega_{n,\mu})|\,dx\,dt\right| \\
&\leq \xi_k(2k)\int_{Q_T}|a_i(x,t,\nabla T_k(u))|\,|D^i T_k(u_n)-D^i T_k(u)|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n\to\infty,
\end{aligned}
\qquad (4.39)
$$

and

$$
\int_{Q_T} a_i(x,t,\nabla T_k(u_n))\,D^i T_k(u)\,|\xi_k(\omega_{n,\mu})|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n \text{ and } \mu\to\infty. \qquad (4.40)
$$

Having in mind (4.37)–(4.40), we conclude that

$$
\begin{aligned}
\left|\mathcal{J}^4_{n,\mu,h}\right| \leq \frac{b(k)}{\alpha}\sum_{i=1}^N\int_{Q_T}&(a_i(x,t,\nabla T_k(u_n)) \\
&-a_i(x,t,\nabla T_k(u)))(D^i T_k(u_n) \\
&-D^i T_k(u))\,|\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&+\varepsilon_6(n,\mu).
\end{aligned}
\qquad (4.41)
$$

The fifth term: We have

$$\begin{aligned}\mathcal{J}^5_{n,\mu,h} &= \int_{\{|u_n|\le k\}} d(x,t)|T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n)-(T_k(u))_\mu)\exp(\gamma\omega^2_{n,\mu})\,dx\,dt \\ &\ge d_0\int_{Q_T}\Big(|T_k(u_n)|^{p_0-2}T_k(u_n)-|T_k(u)|^{p_0-2}T_k(u)\Big)(T_k(u_n) \\ &-T_k(u))\exp(\gamma\omega^2_{n,\mu})\,dx\,dt \\ &+\int_{Q_T} d(x,t)|T_k(u)|^{p_0-2}T_k(u)(T_k(u_n)-T_k(u))\exp(\gamma\omega^2_{n,\mu})\,dx\,dt \\ &+\int_{\{|u_n|\le k\}} d(x,t)|T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u)-(T_k(u))_\mu)\exp(\gamma\omega^2_{n,\mu})\,dx\,dt \\ &-\int_{\{|u_n|>k\}} d(x,t)|T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\gamma\omega^2_{n,\mu})\,dx\,dt.\end{aligned}$$

Since $T_k(u_n)-T_k(u)\to 0$ and $T_k(u)-(T_k(u))_\mu\to 0$ strongly in $L^{p_0}(Q_T)$, then the three last terms on the right-hand side tends to 0, and we obtain

$$\begin{aligned}\mathcal{J}^5_{n,\mu,h} \ge d_0\int_{Q_T}&\big(|T_k(u_n)|^{p_0-2}T_k(u_n)-|T_k(u)|^{p_0-2}T_k(u)\big) \\ &(T_k(u_n)-T_k(u))\,dx\,dt+\varepsilon_7(n,\mu).\end{aligned} \tag{4.42}$$

The sixth term:
We have $f_n\to f$ in $L^1(Q_T)$, and since $\xi_k(\omega_{n,\mu})\rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, then

$$\left|\mathcal{J}^6_{n,\mu,h}\right| \le \|S'_h\|_{L^\infty(I\!R)}\int_{Q_T}|f_n|\,|\xi_k(T_k(u_n)-(T_k(u))_\mu)|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,\ \mu\to 0. \tag{4.43}$$

The seventh and last terms: Let n be large enough, it's clear that $\phi_{i,n}(T_{h+1}(u_n)) = \phi_i(T_{h+1}(u_n))\to\phi_i(T_{h+1}(u))$ in $L^{p'_i}(Q_T)$, and since $D^iT_k(u_n)-D^i(T_k(u))_\mu\rightharpoonup 0$ in $L^{p_i}(Q_T)$, we conclude that

$$\begin{aligned}\left|\mathcal{J}^7_{n,\mu,h}\right| &\le \sum_{i=1}^N\Big|\int_{\{|u_n|\le h+1\}}\phi_{i,n}(T_{h+1}(u_n))(D^iT_k(u_n) \\ &-D^i(T_k(u))_\mu)S'_h(u_n)\xi'_k(\omega_{n,\mu})\,dx\,dt\Big| \\ &\le \xi'_k(2k)\|S'_h\|_\infty\sum_{i=1}^N\int_{\{|u_n|\le h+1\}}|\phi_i(T_{h+1}(u_n))|\,|D^iT_k(u_n) \\ &-D^i(T_k(u))_\mu|\,dx\,dt\to 0,\end{aligned} \tag{4.44}$$

as n and μ tend to infinity. Concerning the last term, in view of Young's inequality and (4.24), we obtain

$$
\begin{aligned}
\left|\mathcal{J}^8_{n,\mu,h}\right| &\leq \sum_{i=1}^{N}\left|\int_{\{h<|u_n|\leq h+1\}} \phi_{i,n}(T_{h+1}(u_n))D^i u_n S''_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt\right| \\
&\leq \|S''_h\|_{L^\infty(I\!R)}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} |\phi_{i,n}(T_{h+1}(u_n))||D^i u_n||\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&\leq \|S''_h\|_{L^\infty(I\!R)}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} \frac{|\phi_{i,n}(T_{h+1}(u_n))|^{p'_i}}{p'_i}|\xi_k(\omega_{n,\mu})|\,dx\,dt \\
&+\xi_k(2k)\|S''_h\|_{L^\infty(I\!R)}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} \frac{|D^i u_n|^{p_i}}{p_i}\,dx\,dt \longrightarrow 0 \\
&\qquad \text{as}\quad n,\ \mu \text{ then } h\to\infty.
\end{aligned}
\tag{4.45}
$$

Combining (4.29), (4.35), (4.36) and (4.41)–(4.45), we deduce that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{Q_T}\Big(a_i(x,t,\nabla T_k(u_n))-a_i(x,t,\nabla T_k(u))\Big)(D^i T_k(u_n) \\
&-D^i T_k(u))\Big(\xi'_k(\omega_{n,\mu})-\frac{b(k)}{\alpha}|\xi_k(\omega_{n,\mu})|\Big)\,dx\,dt \\
&+d_0\int_{Q_T}(|T_k(u_n)|^{p_0-2}T_k(u_n) \\
&-|T_k(u)|^{p_0-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\,dt \\
&\leq \varepsilon_9(n,\mu,h).
\end{aligned}
$$

By letting n,μ and h tend to infinity, we get

$$
\begin{aligned}
\lim_{n\to\infty}&\sum_{i=1}^{N}\int_{Q_T}\big(a_i(x,t,\nabla T_k(u_n))-a_i(x,t,\nabla T_k(u))\big)(D^i T_k(u_n)-D^i T_k(u))\,dx\,dt \\
&+d_0\int_{Q_T}(|T_k(u_n)|^{p_0-2}T_k(u_n)-|T_k(u)|^{p_0-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\,dt=0.
\end{aligned}
$$

In view of Lemma 3.3, we obtain

$$
T_k(u_n)\longrightarrow T_k(u)\quad\text{in}\quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))\quad\text{and}\quad D^i u_n\longrightarrow D^i u\quad\text{a.e in } Q_T.
\tag{4.46}
$$

Step 5 : The equi-integrability of $(g_n(x,t,u_n,\nabla u_n))_n$ **and** $(|u_n|^{p_0-2}u_n)_n$. We prove that

$$
\begin{aligned}
&g_n(x,t,u_n,\nabla u_n)\longrightarrow g(x,t,u,\nabla u)\quad\text{and} \\
&|u_n|^{p_0-2}u_n\longrightarrow |u|^{p_0-2}u\quad\text{strongly in}\quad L^1(Q_T).
\end{aligned}
$$

Using Vitali's theorem, it suffices to prove that the sequences $(g_n(x,t,u_n,\nabla u_n))_n$ and $(|u_n|^{p_0-2}u_n)_n$ are uniformly equi-integrable. Indeed, taking $T_{h+1}(u_n)-T_h(u_n)$ as a test function in (4.2), we have

$$\begin{aligned}
&\int_{Q_T}\frac{\partial\varphi_{h+1}(u_n)}{\partial t}-\frac{\partial\varphi_h(u_n)}{\partial t}\,dt\,dx+\sum_{i=1}^{N}\int_{\{h\le|u_n|\le h+1\}}a_i(x,t,\nabla u_n)\,D^iu_n\,dx\,dt\\
&\quad+\int_{\{h\le|u_n|\}}g_n(x,t,u_n,\nabla u_n)(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\quad+\int_{\{h\le|u_n|\}}d(x,t)|u_n|^{p_0-2}u_n(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\quad=\int_{\{h\le|u_n|\}}f_n\cdot(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\quad+\sum_{i=1}^{N}\int_{Q_T}\phi_{i,n}(u_n)(D^iT_{h+1}(u_n)-D^iT_h(u_n))\,dx\,dt.
\end{aligned}\tag{4.47}$$

We also have

$$\begin{aligned}
\int_{Q_T}\frac{\partial\varphi_{h+1}(u_n)}{\partial t}-\frac{\partial\varphi_h(u_n)}{\partial t}\,dt\,dx&=\int_\Omega\varphi_{h+1}(u_n(T))\,dx-\int_\Omega\varphi_{h+1}(u_{0,n})\,dx\\
&\quad-\int_\Omega\varphi_h(u_n(T))\,dx+\int_\Omega\varphi_h(u_{0,n})\,dx.
\end{aligned}$$

Now by (4.19), we get

$$\int_{Q_T}\frac{\partial\varphi_{h+1}(u_n)}{\partial t}-\frac{\partial\varphi_h(u_n)}{\partial t}\,dt\,dx\ge-\int_\Omega\varphi_{h+1}(u_{0,n})\,dx+\int_\Omega\varphi_h(u_{0,n})\,dx.$$

Using (1.3) and (4.20), we obtain

$$\begin{aligned}
&\int_{\{h+1\le|u_n|\}}|g_n(x,t,u_n,\nabla u_n)|\,dx\,dt+d_0\int_{\{h+1\le|u_n|\}}|u_n|^{p_0-1}\,dx\,dt\\
&\le\int_{\{h\le|u_n|\}}g_n(x,t,u_n,\nabla u_n)(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\quad+\int_{\{h\le|u_n|\}}d(x,t)|u_n|^{p_0-2}u_n(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\le\int_{\{h\le|u_n|\}}|f|\,dx\,dt+\int_\Omega\varphi_{h+1}(u_{0,n})\,dx-\int_\Omega\varphi_h(u_{0,n})\,dx.
\end{aligned}\tag{4.48}$$

Hence, on one hand, in view of (4.22)–(4.23), we deduce that for all $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)\le|u_n|\}}|g_n(x,t,u_n,\nabla u_n)|\,dx\,dt+d_0\int_{\{h(\eta)\le|u_n|\}}|u_n|^{p_0-1}\,dx\,dt\le\frac{\eta}{2}.\tag{4.49}$$

On the other hand, for any measurable subset $E \subset Q_T$, we have

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + d_0 \int_E |u_n|^{p_0-1}\,dx\,dt \\
&\le b(h(\eta)) \int_E (c(x,t) + \sum_{i=1}^{N} |D^i T_{h(\eta)}(u_n)|^{p_i})\,dx\,dt \\
&+ d_0 \int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\,dt \\
&+ \int_{\{h(\eta)\le |u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \\
&+ d_0 \int_{\{h(\eta)\le |u_n|\}} |u_n|^{p_0-1}\,dx\,dt.
\end{aligned} \tag{4.50}$$

Thanks to (4.46), there exists $\beta(\eta) > 0$ such that

$$\begin{aligned}
&b(h(\eta)) \int_E (c(x,t) + \sum_{i=1}^{N} |D^i T_{h(\eta)}(u_n)|^{p_i})\,dx\,dt + d_0 \int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\,dt \\
&\le \frac{\eta}{2} \quad \text{for} \quad meas(E) \le \beta(\eta).
\end{aligned} \tag{4.51}$$

Finally, by combining (4.49),(4.50) and (4.51), we obtain

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + d_0 \int_E |u_n|^{p_0-1}\,dx\,dt \le \eta, \\
&\qquad \text{with} \quad meas(E) \le \beta(\eta),
\end{aligned} \tag{4.52}$$

which implies that $(g_n(x,t,u_n,\nabla u_n))_n$ and $(|u_n|^{p_0-2}u_n)_n$ are uniformly equi-integrable. Then, in view of Vitali's theorem, we deduce that

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \quad \text{and} \quad |u_n|^{p_0-2}u_n \longrightarrow |u|^{p_0-2}u \quad \text{in} \quad L^1(Q_T). \tag{4.53}$$

Step 6 : The convergence of u_n in $C([0,T];L^1(\Omega))$. Let m and n be two integers, then u_n and u_m verifies

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t} - \frac{\partial u_m}{\partial t}, \psi \rangle\,dt + \sum_{i=1}^{N} \int_{Q_T} \big(a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u_m)\big)\, D^i \psi\,dx\,dt \\
&+ \int_{Q_T} \big(g_n(x,t,u_n,\nabla u_n) - g_m(x,t,u_m,\nabla u_m)\big)\psi\,dx\,dt \\
&+ \int_{Q_T} d(x,t)\big(|u_n|^{p_0-2}u_n - |u_m|^{p_0-2}u_m\big)\psi\,dx\,dt \\
&= \int_{Q_T} (f_n - f_m)\psi\,dx\,dt + \sum_{i=1}^{N} \int_{Q_T} \big(\phi_{i,n}(u_n) - \phi_{i,m}(u_m)\big)\, D^i \psi\,dx\,dt,
\end{aligned}$$

for all $\psi \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$. By taking $\psi = T_1(u_n - u_m)\cdot\chi_{[0,s]}$ for $0 < s \le T$, we obtain

$$\begin{aligned}
&\int_\Omega \int_0^s \frac{\partial \varphi_1(u_n - u_m)}{\partial t}\,dt\,dx \\
&\quad + \sum_{i=1}^N \int_0^s \int_\Omega \big(a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u_m)\big)\, D^i T_1(u_n - u_m)\,dx\,dt \\
&\quad + \int_0^s \int_\Omega \big(g_n(x,t,u_n,\nabla u_n) - g_m(x,t,u_m,\nabla u_m)\big) T_1(u_n - u_m)\,dx\,dt \\
&\quad + \int_0^s \int_\Omega d(x,t)\big(|u_n|^{p_0-2}u_n - |u_m|^{p_0-2}u_m\big) T_1(u_n - u_m)\,dx\,dt \\
&\quad = \int_0^s \int_\Omega (f_n - f_m) T_1(u_n - u_m)\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} \big(\phi_{i,n}(u_n) - \phi_{i,m}(u_m)\big)\, D^i T_1(u_n - u_m)\,dx\,dt.
\end{aligned} \tag{4.54}$$

Note that we have

$$\int_\Omega \int_0^s \frac{\partial \varphi_1(u_n - u_m)}{\partial t}\,dt\,dx = \int_\Omega \varphi_1(u_n(s) - u_m(s))\,dx - \int_\Omega \varphi_1(u_{0,n} - u_{0,m})\,dx.$$

Concerning the terms on the left-hand side of (4.54), it's clear that $D^i T_1(u_n - u_m) = (D^i u_n - D^i u_m).\chi_{\{|u_n - u_m| \le 1\}}$. Then

$$\sum_{i=1}^N \int_0^s \int_\Omega \Big(a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u_m)\Big) D^i T_1(u_n - u_m)\,dx\,dt \ge 0. \tag{4.55}$$

In view of (4.53), we get

$$\begin{aligned}
0 &\le \int_0^s \int_\Omega d(x,t)\Big(|u_n|^{p_0-2}u_n - |u_m|^{p_0-2}u_m\Big) T_1(u_n - u_m)\,dx\,dt \\
&\le \|d(\cdot)\|_{L^\infty(Q_T)} \int_{Q_T} \Big||u_n|^{p_0-2}u_n - |u_m|^{p_0-2}u_m\Big|\,dx\,dt \longrightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_0^s \int_\Omega (g_n(x,t,u_n,\nabla u_n) - g_m(x,t,u_m,\nabla u_m)) T_1(u_n - u_m)\,dx\,dt\Big| \\
&\le \int_{Q_T} |g_n(x,t,u_n,\nabla u_n) - g_m(x,t,u_m,\nabla u_m)|\,dx\,dt \longrightarrow 0,
\end{aligned}$$

as n and m tend to infinity. For the terms on the right-hand side of (4.54), we have

$$\Big| \int_0^s \int_\Omega (f_n - f_m) \cdot T_1(u_n - u_m)\, dx\, dt \Big| \leq \int_{Q_T} |f_n - f_m|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

Also, we prove that (see Appendix)

$$\sum_{i=1}^N \int_{Q_T} \big(\phi_{i,n}(u_n) - \phi_{i,m}(u_m)\big)\, D^i T_1(u_n - u_m)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

Now since $\varphi_1(u_{0,n} - u_{0,m}) \to 0$ in $L^1(\Omega)$, then

$$\int_\Omega \varphi_1(u_n(s) - u_m(s))\, dx \longrightarrow 0 \quad \text{as} \quad n, m \to \infty. \tag{4.56}$$

We also have

$$\int_{\{|u_n - u_m| \leq 1\}} |u_n(s) - u_m(s)|^2\, dx + \int_{\{|u_n - u_m| > 1\}} |u_n(s) - u_m(s)|\, dx \leq 2 \int_\Omega \varphi_1(u_n(s) - u_m(s))\, dx, \tag{4.57}$$

and

$$\begin{aligned} \int_\Omega |u_n(s) - u_m(s)|\, dx &= \int_{\{|u_n - u_m| \leq 1\}} |u_n(s) - u_m(s)|\, dx \\ &\quad + \int_{\{|u_n - u_m| > 1\}} |u_n(s) - u_m(s)|\, dx \\ &\leq \Big(\int_{\{|u_n - u_m| \leq 1\}} |u_n(s) - u_m(s)|^2\, dx \Big)^{\frac{1}{2}} \cdot (\text{meas}(\Omega))^{\frac{1}{2}} \\ &\quad + \int_{\{|u_n - u_m| > 1\}} |u_n(s) - u_m(s)|\, dx. \end{aligned} \tag{4.58}$$

Hence in view of (4.56)–(4.58), we deduce that

$$\int_\Omega |u_n(s) - u_m(s)|\, dx \longrightarrow 0 \quad \text{as} \quad m, n \to \infty. \tag{4.59}$$

Thus $(u_n)_n$ is a Cauchy sequence in $C([0, T]; L^1(\Omega))$. Therefore u_n converges to $u \in C([0, T]; L^1(\Omega))$ and we have $u_n(s) \to u(s)$ in $L^1(\Omega)$ for any $0 \leq s \leq T$.

Step 7 : Passage to the limit. Let $\psi \in L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$ with $\dfrac{\partial \psi}{\partial t} \in L^{\vec{p}'}(0, T; W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$, and $M = k + \|\psi\|_{L^\infty(Q_T)}$ with $k > 0$.

Choose $T_k(u_n - \psi)$ as a test function in (4.2), we obtain

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n - \psi)\rangle \, dt + \sum_{i=1}^N \int_{Q_T} a_i(x, t, \nabla u_n) \, D^i T_k(u_n - \psi) \, dx \, dt$$
$$+ \int_{Q_T} g_n(x, t, u_n, \nabla u_n) T_k(u_n - \psi) \, dx \, dt$$
$$+ \int_{Q_T} d(x, t)|u_n|^{p_0 - 2} u_n T_k(u_n - \psi) \, dx \, dt$$
$$= \int_{Q_T} f_n T_k(u_n - \psi) \, dx \, dt + \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(u_n) \, D^i T_k(u_n - \psi) \, dx \, dt. \tag{4.60}$$

If $|u_n| > M$, then $|u_n - \psi| \geq |u_n| - \|\psi\|_\infty > k$. Therefore $\{|u_n - \psi| \leq k\} \subseteq \{|u_n| \leq M\}$, which implies that

$$\sum_{i=1}^N \int_{Q_T} a_i(x, t, \nabla u_n) \, D^i T_k(u_n - \psi) \, dx \, dt$$
$$= \sum_{i=1}^N \int_{\{|u_n - \psi| \leq k\}} a_i(x, t, \nabla T_M(u_n))(D^i T_M(u_n) - D^i \psi) \, dx \, dt$$
$$= \sum_{i=1}^N \int_{\{|u_n - \psi| \leq k\}} (a_i(x, t, \nabla T_M(u_n)) - a_i(x, t, \nabla \psi))(D^i T_M(u_n) - D^i \psi) \, dx \, dt$$
$$+ \sum_{i=1}^N \int_{\{|u_n - \psi| \leq k\}} a_i(x, t, \nabla \psi)(D^i T_M(u_n) - D^i \psi) \, dx \, dt.$$

On one hand, since $D^i T_M(u_n) \to D^i T_M(u)$ in $L^{p_i}(Q_T)$, and in view of Fatou's lemma, we obtain

$$\liminf_{n \to +\infty} \sum_{i=1}^N \int_{Q_T} a_i(x, t, \nabla u_n) \, D^i T_k(u_n - \psi) \, dx \, dt$$
$$\geq \sum_{i=1}^N \int_{\{|u - \psi| \leq k\}} (a_i(x, t, \nabla T_M(u)) - a_i(x, t, \nabla \psi))(D^i T_M(u) - D^i \psi) \, dx \, dt$$
$$+ \sum_{i=1}^N \int_{\{|u - \psi| \leq k\}} a_i(x, t, \nabla \psi)(D^i T_M(u) - D^i \psi) \, dx \, dt$$
$$= \sum_{i=1}^N \int_{Q_T} a_i(x, t, \nabla u) \, D^i T_k(u - \psi) \, dx \, dt. \tag{4.61}$$

On the other hand, for the first term on the left-hand side of (4.60), we have

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n-\psi)\rangle \,dt\\&= \int_0^T \langle \frac{\partial (u_n-\psi)}{\partial t}, T_k(u_n-\psi)\rangle \,dt + \int_0^T \langle \frac{\partial \psi}{\partial t}, T_k(u_n-\psi)\rangle \,dt\\&= \int_\Omega \varphi_k(u_n(T)-\psi(T))\,dx - \int_\Omega \varphi_k(u_{0,n}-\psi(0))\,dx\\&\quad + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n-\psi)\,dx\,dt.\end{aligned}$$

Now since $u_n \to u$ in $C([0,T];L^1(\Omega))$, then $u_n(T) \to u(T)$ in $L^1(\Omega)$. It follows that

$$\begin{aligned}&\int_\Omega \varphi_k(u_{0,n}-\psi(0))\,dx \longrightarrow \int_\Omega \varphi_k(u_0-\psi(0))\,dx \quad \text{and}\\&\int_\Omega \varphi_k(u_n(T)-\psi(T))\,dx \longrightarrow \int_\Omega \varphi_k(u(T)-\psi(T))\,dx,\end{aligned} \tag{4.62}$$

as n tends to infinity. Note that we have $\frac{\partial \psi}{\partial t} \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$, and since $T_k(u_n-\psi) \rightharpoonup T_k(u-\psi)$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and weak$-\star$ in $L^\infty(Q_T)$, then

$$\int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\,dx\,dt, \tag{4.63}$$

and

$$\int_{Q_T} f_n T_k(u_n-\psi)\,dx\,dt \to \int_{Q_T} f\, T_k(u-\psi)\,dx\,dt. \tag{4.64}$$

Thanks to (4.53), we deduce that

$$\begin{aligned}&\int_{Q_T} d(x,t)|u_n|^{p_0-2}u_n T_k(u_n-\psi)\,dx\,dt\\&\quad\longrightarrow \int_{Q_T} d(x,t)|u|^{p_0-2}u\; T_k(u-\psi)\,dx\,dt,\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}&\int_{Q_T} g_n(x,t,u_n,\nabla u_n)T_k(u_n-\psi)\,dx\,dt\\&\quad\longrightarrow \int_{Q_T} g(x,t,u,\nabla u)T_k(u-\psi)\,dx\,dt.\end{aligned} \tag{4.66}$$

Since $\phi_{i,n}(T_M(u_n)) = \phi_i(T_M(u_n))$ for n large enough ($n \geq M$), then

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T}\phi_{i,n}(u_n)D^iT_k(u_n-\psi)\,dx\,dt\\
&\quad=\lim_{n\to\infty}\int_{\{|u_n-\psi|\leq k\}}\phi_i(T_M(u_n))(D^iT_M(u_n)-D^i\psi)\,dx\,dt\\
&\quad=\int_{\{|u-\psi|\leq k\}}\phi_i(T_M(u))(D^iT_M(u)-D^i\psi)\,dx\,dt\\
&\quad=\int_{Q_T}\phi_i(u)D^iT_k(u-\psi)\,dx\,dt.
\end{aligned}
\tag{4.67}
$$

By combining (4.60)–(4.66), we deduce that

$$
\begin{aligned}
&\int_\Omega\varphi_k(u(T)-\psi(T))\,dx-\int_\Omega\varphi_k(u_0-\psi(0))\,dx\\
&+\int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u-\psi)\,dx\,dt\\
&+\sum_{i=1}^N\int_{Q_T}a_i(x,t,\nabla u)\,D^iT_k(u-\psi)\,dx\,dt\\
&+\int_{Q_T}g(x,t,u,\nabla u)T_k(u-\psi)\,dx\,dt\\
&+\int_{Q_T}d(x,t)|u|^{p_0-2}u\,T_k(u-\psi)\,dx\,dt\leq\int_{Q_T}fT_k(u-\psi)\,dx\,dt\\
&+\sum_{i=1}^N\int_{Q_T}\phi_i(u)D^iT_k(u-\psi)\,dx\,dt.
\end{aligned}
$$

This concludes our proof.

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5 Appendix

Lemma 5.1 *The operator* $B_n = A + G_n$ *is pseudo-monotone acted from* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *into* $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$. *Moreover,* B_n *is coercive in the following sense*

$$\frac{\int_0^T \langle B_n v, v\rangle\, dt}{\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow +\infty \quad as$$
$$\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \longrightarrow +\infty \quad for \;\; v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)).$$

Proof Using the Hölder's inequality and the growth condition (1.2) we can show that the operator A is bounded, and by using (4.3), we conclude that B_n is bounded. For the coercivity, we have for any $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$,

$$\begin{aligned}
\int_0^T \langle B_n u, u\rangle\, dt &= \int_0^T \langle Au, u\rangle\, dt + \int_0^T \langle G_n u, u\rangle\, dt \\
&= \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u)\, D^i u\, dx\, dt \\
&\quad + \int_{Q_T} d(x,t)|u|^{p_0}\, dx\, dt + \int_{Q_T} g_n(x,t,u,\nabla u) u\, dx\, dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} \phi_n(u)\, D^i u\, dx\, dt \\
&\geq \alpha \sum_{i=1}^N \int_{Q_T} |D^i u|^{p_i}\, dx\, dt + d_0 \int_{Q_T} |u|^{p_0}\, dx\, dt \\
&\quad -2 \sum_{i=1}^N \|\phi(T_n(u))\|_{L^{p_i'}(Q_T)}\, \|D^i u\|_{L^{p_i}(Q_T)} \\
&\geq \alpha \sum_{i=1}^N (\|D^i u\|^{\underline{p}}_{L^{p_i}(Q_T)} - 1) \\
&\quad + d_0(\|u\|^{\underline{p}}_{L^{p_0}(Q_T)} - 1) - C_4\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \\
&\geq C_5 \|u\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} - \alpha N - d_0 - C_4\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))},
\end{aligned}$$

which implies that

$$\frac{\int_0^T \langle B_n u, u\rangle\, dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow +\infty \quad as \quad \|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \to +\infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \\ B_n u_k \rightharpoonup \chi_n & \text{in } L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)), \\ \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{5.1}$$

We prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \text{ as } k \to +\infty.$$

Using the compact embedding (2.5), we have $u_k \to u$ in $L^1(Q_T)$ for a subsequence still denoted $(u_k)_k$.

We also have $(u_k)_k$ is a bounded sequence in $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$, then by the growth condition $(a_i(x, t, \nabla u_k))_k$ is bounded in $L^{p_i'}(Q_T)$. Therefore, there exists a function $\vartheta_i \in L^{p_i'}(Q_T)$ such that

$$a_i(x, t, \nabla u_k) \rightharpoonup \vartheta_i \quad \text{in} \quad L^{p_i'}(Q_T) \quad \text{for} \quad i = 1, \ldots, N, \tag{5.2}$$

and

$$|u_k|^{p_0-2}u_k \rightharpoonup |u|^{p_0-2}u \qquad \text{in} \quad L^{p_0'}(Q_T). \tag{5.3}$$

Similarly, we have $(g_n(x, t, u_k, \nabla u_k))_k$ is bounded in $L^{\underline{p}'}(Q_T)$, then there exists a function $\psi_n \in L^{\underline{p}'}(Q_T)$ such that

$$g_n(x, t, u_k, \nabla u_k) \rightharpoonup \psi_n \text{ in } \quad L^{\underline{p}'}(Q_T) \text{ as } k \to \infty, \tag{5.4}$$

and since $\phi_{i,n} = \phi_i \circ T_n$ is a bounded continuous function, using the Lebesgue dominated convergence theorem, and since $T_n(u_k) \to T_n(u)$ a.e in Q_T, then we get

$$\phi_{i,n}(u_k) \longrightarrow \phi_{i,n}(u) \text{ for } \quad i = 1, \ldots, N. \tag{5.5}$$

On one hand, we have

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^{N} \int_{Q_T} a_i(x, t, \nabla u_k)\, D^i v \, dx\, dt + \lim_{k\to\infty} \int_{Q_T} g_n(x, t, u_k, \nabla u_k) v \, dx\, dt \\
&\quad + \lim_{k\to\infty} \int_{Q_T} d(x, t)|u_k|^{p_0-2} u_k v \, dx\, dt + \lim_{k\to\infty} \sum_{i=1}^{N} \int_{Q_T} \phi_{i,n}(u_k)\, D^i v \, dx\, dt \\
&= \sum_{i=1}^{N} \int_{Q_T} \vartheta_i\, D^i v \, dx\, dt + \int_{Q_T} \psi_n v \, dx\, dt + \int_{Q_T} d(x, t)|u|^{p_0-2} u v \, dx\, dt \\
&\quad + \sum_{i=1}^{N} \int_{Q_T} \phi_{i,n}(u)\, D^i v \, dx\, dt,
\end{aligned} \tag{5.6}$$

for all $v \in L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$. By using (5.1) and (5.6), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\langle B_n(u_k), u_k\rangle \\
&= \limsup_{k\to\infty}\Big\{\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)\ D^i u_k\, dx\, dt + \int_{Q_T} g_n(x,t,u_k,\nabla u_k)u_k\, dx\, dt \\
&+ \int_{Q_T} d(x,t)|u_k|^{p_i}\, dx\, dt + \sum_{i=1}^{N}\int_{Q_T}\phi_i(u_k)\ D^i u_k\, dx\, dt\Big\} \\
&\le \sum_{i=1}^{N}\int_{Q_T}\vartheta_i\ D^i u\, dx\, dt + \int_{Q_T}\psi_n u\, dx\, dt + \int_{Q_T} d(x,t)|u|^{p_i}\, dx\, dt \\
&+ \sum_{i=1}^{N}\int_{Q_T}\phi_{i,n}(u)\ D^i u\, dx\, dt.
\end{aligned} \tag{5.7}$$

Thanks to (5.4) and (5.5), we have

$$\begin{aligned}
&\int_{Q_T} g_n(x,t,u_k,\nabla u_k)u_k\, dx\, dt \longrightarrow \int_{Q_T}\psi_n u\, dx\, dt \quad \text{and} \\
&\int_{Q_T}\phi_i(u_k)\ D^i u_k\, dx\, dt \longrightarrow \int_{Q_T}\phi_i(u)\ D^i u\, dx\, dt.
\end{aligned} \tag{5.8}$$

Therefore

$$\begin{aligned}
&\limsup_{k\to\infty}\Big\{\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)\ D^i u_k\, dx\, dt + \int_{Q_T} d(x,t)|u_k|^{p_0}\, dx\, dt\Big\} \\
&\le \sum_{i=1}^{N}\int_{Q_T}\vartheta_i\ D^i u\, dx\, dt + \int_{Q_T} d(x,t)|u|^{p_0}\, dx\, dt.
\end{aligned} \tag{5.9}$$

On the other hand, using (1.4) we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,\nabla u_k) - a_i(x,t,\nabla u))(D^i u_k - D^i u)\, dx\, dt \\
&+ \int_{Q_T} d(x,t)(|u_k|^{p_0-2}u_k - |u|^{p_0-2}u)(u_k - u)dx\, dt \ge 0,
\end{aligned} \tag{5.10}$$

which gives

$$\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)\ D^i u_k\,dx\,dt+\int_{Q_T} d(x,t)|u_k|^{p_0}dx\,dt$$
$$\geq \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)\ D^i u\,dx\,dt+\int_{Q_T} d(x,t)|u_k|^{p_0-2}u_k u\,dx\,dt$$
$$+\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u)(D^i u_k-D^i u)\,dx\,dt$$
$$+\int_{Q_T} d(x,t)|u|^{p_0-2}u(u_k-u)\,dx\,dt.$$

In view of (5.2) and (5.3), we get

$$\liminf_{k\to\infty}\left\{\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)\ D^i u_k\,dx\,dt+\int_{Q_T} d(x,t)|u_k|^{p_0}\,dx\,dt\right\}$$
$$\geq \sum_{i=1}^{N}\int_{Q_T}\vartheta_i\ D^i u\ \ dx\,dt+\int_{Q_T} d(x,t)|u|^{p_0}\,dx\,dt.$$

This implies, thanks to (5.9), that

$$\lim_{k\to\infty}\left\{\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)\ D^i u_k\,dx\,dt+\int_{Q_T} d(x,t)|u_k|^{p_0}\,dx\,dt\right\}$$
$$=\sum_{i=1}^{N}\int_{Q_T}\vartheta_i\ D^i u\,dx\,dt+\int_{Q_T} d(x,t)|u|^{p_0}\,dx\,dt. \tag{5.11}$$

Using (5.8), we conclude that $\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k\to+\infty$.
Now, by (5.11) we have

$$\lim_{k\to+\infty}\left\{\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,\nabla u_k)-a_i(x,t,\nabla u))(D^i u_k-D^i u)\,dx\,dt\right.$$
$$\left.+d_0\int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)dx\,dt\right\}=0.$$

In view of Lemma 3.3, we get

$$u_k\longrightarrow u\quad \text{in}\ \ L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))\ \ \text{then}\ \ D^i u_k\longrightarrow D^i u\ \ \text{a.e in}\ \ Q_T.$$

It follows that

$$a_i(x,t,\nabla u_k)\rightharpoonup a_i(x,t,\nabla u)\quad \text{in}\ \ L^{p_i'}(Q_T),$$

and

$$g_n(x,t,u_k,\nabla u_k) \rightharpoonup g_n(x,t,u,\nabla u) \quad \text{in} \quad L^{p_i'}(Q_T).$$

We deduce that $\chi_n = B_n u$, which completes the proof of Lemma 5.1. □

Proof of the convergence (4.56).

Let $h > 0$ and n, m large enough, we have

$$\begin{aligned}
&\Big|\int_0^s\int_\Omega \big(\phi_{i,n}(u_n)-\phi_{i,m}(u_m)\big)D^iT_1(u_n-u_m)\,dx\,dt\Big| \\
&\quad\le \int_{\{|u_n|\le h\}\cap\{|u_m|\le h\}} |\phi_i(T_h(u_n))-\phi_i(T_h(u_m))|\,|D^iT_h(u_n)-D^iT_h(u_m)|\,dx\,dt \\
&\quad + \int_{\{|u_n|>h\}\cup\{|u_m|>h\}} |\phi_{i,n}(u_n)-\phi_{i,m}(u_m)|\,|D^iu_n-D^iu_m|.\chi_{\{|u_n-u_m|\le 1\}}\,dx\,dt.
\end{aligned} \tag{5.12}$$

Regarding the first term on the right-hand side of (5.12), since $D^iT_h(u_n)$ and $D^iT_h(u_m)$ converge strongly to $D^iT_h(u)$ in $L^{p_i}(Q_T)$, and since $|\phi_i(T_h(u_n))-\phi_i(T_h(u_m))|$ is bounded in $L^{p_i'}(Q_T)$, then

$$\begin{aligned}
&\int_{\{|u_n|\le h\}\cap\{|u_m|\le h\}} |\phi_i(T_h(u_n)) \\
&\quad -\phi_i(T_h(u_m))|\,|D^iT_h(u_n)-D^iT_h(u_m)|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad m,n\longrightarrow\infty.
\end{aligned} \tag{5.13}$$

Concerning the second term, we have $\phi_{i,n}(\cdot)$ is a continuous function, then there exists $M_1 > 0$ such that $\sup\limits_{|r-s|\le 1} |\phi_{i,n}(s)-\phi_{i,n}(r)| \le M_1$. Also, it's clear that

$$\begin{aligned}
&\forall s \in I\!R \quad \text{and} \quad \forall M_2 > 0 \qquad \text{we have} \qquad |\phi_{i,n}(s)-\phi_{i,m}(s)| \le M_2 \\
&\quad \text{for} \quad n,m \ge n_0(s,M_2).
\end{aligned}$$

Taking n and m large enough, by using (4.12), (4.24) and Young's inequality, we obtain

$$\begin{aligned}
&\int_{\{|u_n|>h\}\cup\{|u_m|>h\}} |\phi_{i,n}(u_n)-\phi_{i,m}(u_m)|\,|D^iu_n-D^iu_m|.\chi_{\{|u_n-u_m|\le 1\}}\,dx\,dt \\
&\le \int_{\{|u_n|>h\}\cup\{|u_m|>h\}} |\phi_{i,n}(u_n)-\phi_{i,n}(u_m)|^{p_i'}\,.\chi_{\{|u_n-u_m|\le 1\}}\,dx\,dt \\
&+ \int_{\{|u_n|>h\}\cup\{|u_m|>h\}} |\phi_{i,n}(u_m)-\phi_{i,m}(u_m)|^{p_i'}\,.\chi_{\{|u_n-u_m|\le 1\}}\,dx\,dt \\
&+ 2\int_{\{|u_n|>h\}\cup\{|u_m|>h\}} |D^iu_n-D^iu_m|^{p_i}.\chi_{\{|u_n-u_m|\le 1\}}\,dx\,dt
\end{aligned}$$

$$\leq (M_1^{p_i'} + M_2^{p_i'})\text{meas}\,(\{|u_n| > h\} \cup \{|u_m| > h\})$$
$$+ 2^{p_i} \int_{\{|u_n|>h-1\}\cap\{|u_n|-1<|u_m|\leq|u_n|+1\}} |D^i u_m|^{p_i}\, dx\, dt$$
$$+ 2^{p_i} \int_{\{|u_m|>h-1\}\cap\{|u_m|-1<|u_n|\leq|u_m|+1\}} |D^i u_n|^{p_i}\, dx\, dt \longrightarrow 0 \quad \text{as} \quad h \to \infty. \tag{5.14}$$

Thus, by combining (5.12)–(5.14), we get

$$\int_0^s \int_\Omega \big(\phi_{i,n}(u_n) - \phi_{i,m}(u_m)\big) D^i T_1(u_n - u_m)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n, m \to \infty. \tag{5.15}$$

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